Pseudospectra and Matrix Behavior

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Abstract

Whereas the spectrum of a normal matrix determines its behavior, the pseudospectrum offers an alternative to better understand the behavior of matrices that are nonnormal. In this study, we investigate the relationship between pseudospectra and matrix behavior. In particular, we investigate the implications of matrices with equal spectral norm pseudospectra.

Notation/Definitions

Throughout, $\mathbb{M}_n$ denotes the set of n-by-n matrices with complex entries and $\Lambda(A)$ denotes the set of eigenvalues of $A \in \mathbb{M}_n$.

ϵ-pseudospectra

For $A \in \mathbb{M}_n$ and $\epsilon > 0$, the ϵ-*pseudospectrum* of A is defined by

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\| \geqslant \epsilon^{-1}\}$$

Spectral Norm

For $A \in \mathbb{M}_n$, the *spectral norm* of A is defined by

$$\|A\| = \sqrt{\lambda_{\max}}$$

where $\lambda_{\max}$ is the largest number λ such that $A^*A - \lambda I$ is singular.

Minimal and Characteristic Polynomials

For $A \in \mathbb{M}_n$, the *minimal polynomial* of A, denoted by m_A, is the monic polynomial of minimal degree so that $m_A(A) = 0$. The *characteristic polynomial* of A, denoted by χ_A, in variable λ, is the determinant of $(A - \lambda I)$ where I is the identity matrix in $\mathbb{M}_n$. It is well-known that the minimal and characteristic polynomials have the same zeros.

Background

In [1], Greenbaum and Trefethen showed that square matrices with equal spectral norm pseudospectra does not imply that they have the same behavior. In other words, there are square matrices A and B such that, under the spectral norm,

$$\|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad \forall z \in \mathbb{C} \tag{R}$$

does *not* imply

$$\|p(A)\| = \|p(B)\| \quad \text{for all polynomials } p \tag{P}$$

To show that $(R) \not\Longrightarrow (P)$, their example consisted of 5-by-5 block diagonal matrices using Jordan blocks. To better understand the relationship between (R) and (P) in the spectral norm, we first prove the simple case that if the pseudospectra of 2×2 matrices match, then they have the same minimal polynomial.

Given matrices $A, B \in \mathbb{M}_n$, let (M) be the condition $m_A = m_B$. We show that $(R) \Longrightarrow (M)$, and then show one case in which $(R) \Longrightarrow (P)$.

Useful Lemmas

Lemma 1 Similar matrices have the same minimal polynomial.

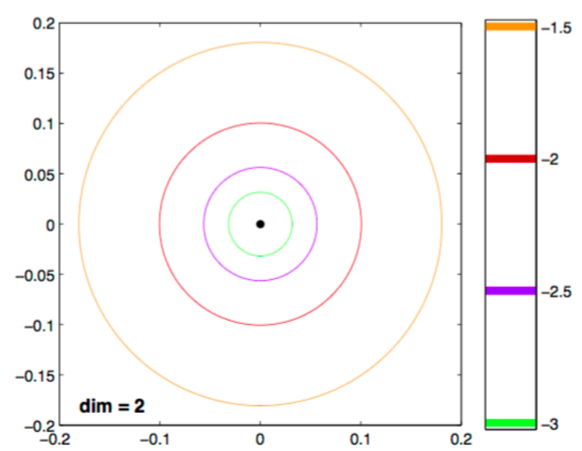
Lemma 2 Let $A, B \in \mathbb{M}_n$ and $U, V \in \mathbb{M}_n$ be unitary, and $A = U^*T_aU$, $B = V^*T_bV$ where the diagonal entries of T_a and T_b are the eigenvalues of A and B, respectively. If condition (R) holds for A and B, then condition (R) holds for T_a and T_b.

Lemma 3

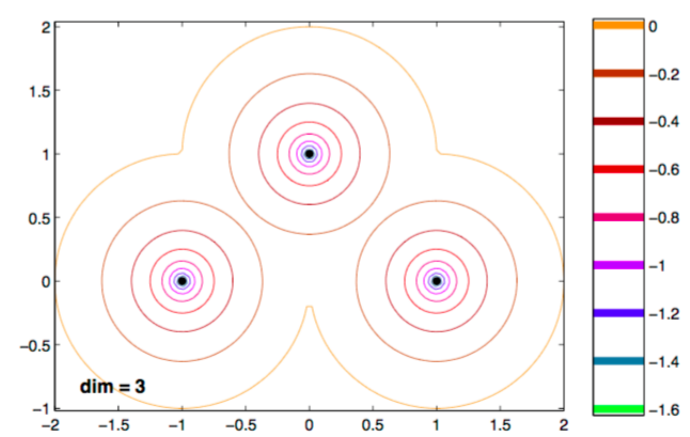
If $T_c = \begin{pmatrix} \lambda & c \\ & \lambda \end{pmatrix}$, then $\|(zI - T_c)^{-1}\| = |z - \lambda|^{-1}\left[\frac{|x|^2}{2} + 1 + \frac{\sqrt{|x|^4 + 4|x|^2}}{2}\right]$ where $x = -c(z - \lambda)^{-1}$.

Lemma 4

Let $T_a = \begin{pmatrix} \lambda & a \\ & \lambda \end{pmatrix}$ and $T_b = \begin{pmatrix} \lambda & b \\ & \lambda \end{pmatrix}$. If $\|(zI - T_a)^{-1}\| = \|(zI - T_b)^{-1}\|$ for all $z \in \mathbb{C}$, then $|a| = |b|$.



Pseudospectra of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$



Pseudospectra of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}$

$(R) \Longrightarrow (M)$.

Let $A, B \in \mathbb{M}_2$ and assume that condition (R) holds. This implies that $\Lambda(A) = \Lambda(B)$. By Schur's Triangularization Theorem, there exists unitary matrices U, V and upper triangular matrices T_a and T_b so that

$$A = U^*T_aU \quad \text{and} \quad B = V^*T_bV$$

where the diagonal entries in T_a and T_b are the eigenvalues of A and B, respectively.

Case 1 $\Lambda(A) = \Lambda(B) = \{\lambda_1, \lambda_2\}$

In this case, we have $T_a = \begin{pmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{pmatrix}$ and $T_b = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}$ for some $a, b \in \mathbb{C}$. Notice that $\chi_{T_a}(\mu) = \chi_{T_b}(\mu) = (\mu - \lambda_1)(\mu - \lambda_2)$. So, the minimal polynomial for T_a and T_b is their characteristic polynomial. Therefore by Lemma 1, condition (M) holds.

Case 2 $\Lambda(A) = \Lambda(B) = \{\lambda\}$

In this case, we have $T_a = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}$ and $T_b = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$ for some $a, b \in \mathbb{C}$. By Lemma 2, $\|(zI - T_a)^{-1}\| = \|(zI - T_b)^{-1}\|$ and so by Lemma 4, $|a| = |b|$. Now, if $a = b = 0$, then A and B are similar to a scalar multiple of the identity matrix, and so $m_A(z) = z - \lambda = m_B(z)$. On the other hand, if a or b is nonzero, then both a and b are nonzero by Lemma 4. Therefore by Lemma 1, $m_A(z) = (z - \lambda)^2 = m_B(z)$. In either case, $m_A = m_B$ and so condition (M) holds. ■

Implications of (M)

We now see that condition (M) becomes very useful as we endeavor to show $(R) \Longrightarrow (P)$. If we fix a polynomial p, and let m_A be the minimal polynomial for a 2×2 matrix A, then the Division Algorithm tells us that there exists polynomials q_A and r_A so that

$$p = m_A \cdot q_A + r_A \quad \text{where} \quad \deg r_A < \deg m_A$$

and so

$$p(A) = m_A(A) \cdot q_A(A) + r_A(A) = r_A(A)$$

which implies that $\|p(A)\| = \|r(A)\|$. Since $\deg r_A < \deg m_A = 2$, then it suffices to show that condition (P) holds for all monic linear polynomials.

$(R) \Longrightarrow (P)$ when $\Lambda(A) = \Lambda(B) = \{\lambda\}$.

Recall from before that $A = U^* \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} U$ and $B = V^* \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} V$ for unitaries U and V. Let $p(t) = t - \mu$ be an arbitrary monic linear polynomial. We need to verify that $\|p(A)\| = \|p(B)\|$. Notice that $T_a = \lambda I + aN$ and $T_b = \lambda I + bN$ where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Now, observe that when $z = \mu - 2\lambda$,

$$\left\|(A + zI)^{-1}\right\| = |\mu - \lambda|^{-1} \cdot \left\|(I + aN/(\mu - \lambda))^{-1}\right\| \quad \text{(and similarly for } \left\|(B + zI)^{-1}\right\|\text{)}$$

By condition (R), $\left\|(I + aN/(\mu - \lambda))^{-1}\right\| = \left\|(I + bN/(\mu - \lambda))^{-1}\right\|$. Note that $(I + xN)^{-1} = I - xN$ for any x. Therefore,

$$\begin{aligned}
\|I - aN/(\mu - \lambda)\| &= \|I - bN/(\mu - \lambda)\| \\
\|-[(\mu - \lambda)I - aN]\| &= \|-[(\mu - \lambda)I - bN]\| \\
\|\lambda I + aN - \mu I\| &= \|\lambda I + bN - \mu I\| \\
\|UAU^* - U(\mu I)U^*\| &= \|VBV^* - V(\mu I)V^*\| \\
\|U(A - \mu I)U^*\| &= \|V(B - \mu I)V^*\| \\
\|A - \mu I\| &= \|B - \mu I\| \\
\|p(A)\| &= \|p(B)\|
\end{aligned}$$

■

References

[1] Greenbaum, Anne., and Trefethen, Lloyd N. *Do the Pseudospectra of a Matrix Determine Its Behavior?*. Cornell University, 1993. Print.

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[4] Jensen, Arne. *Lecture Notes on Spectra and Pseudospectra of Matrices and Operators*, Aalborg University, 2009. (Images from this source)