The Hilbert Space of Dirichlet Series $\mathcal{H}^2$ and its Multipliers

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Introduction

A Dirichlet series is a series of the form

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\quad \text{for } s\in\mathbb{C},$$

where $(a_n)_n \subseteq \mathbb{C}$. For instance, the **Riemann zeta function** is the Dirichlet series

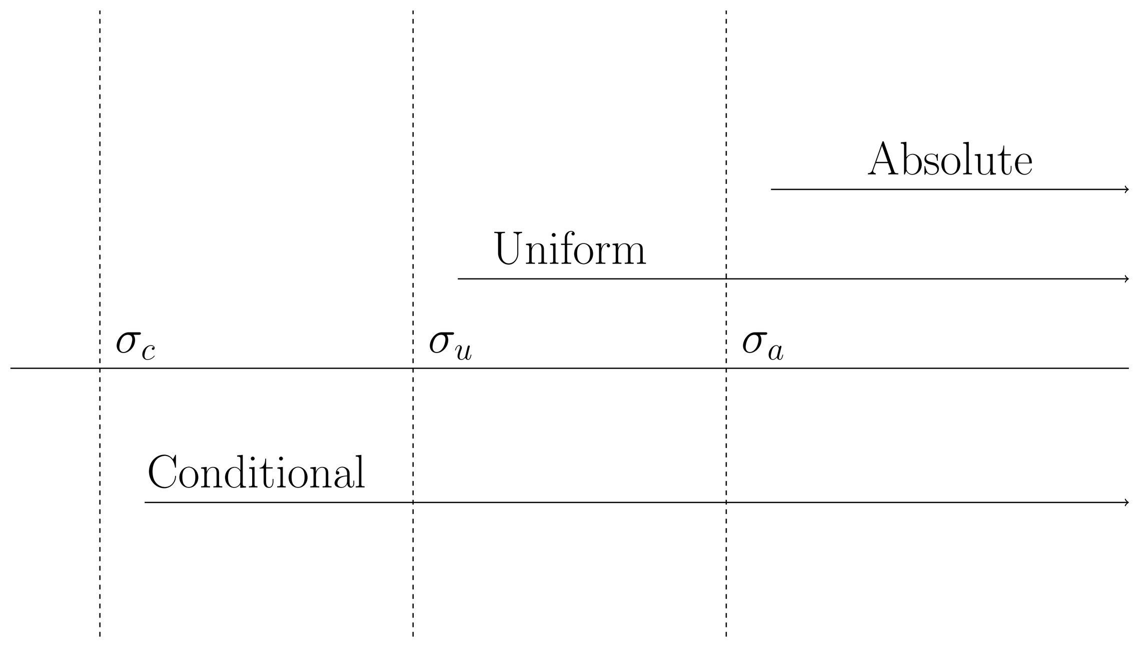
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Clearly, the series on the right diverges when $s=1$. Thus, Dirichlet series need not converge for all $s\in\mathbb{C}$. To describe the different regions of convergence, one defines the following *abscissas of convergence*:

$$\sigma_c=\inf\{r\in\mathbb{R}\mid f \text{ converges for } \operatorname{Re}s>r\},$$
$$\sigma_u=\inf\{r\in\mathbb{R}\mid f \text{ converges uniformly for } \operatorname{Re}s>r\},$$
$$\sigma_b=\inf\{r\in\mathbb{R}\mid f \text{ is analytic and bounded for } \operatorname{Re}s>r\}, \text{ and}$$
$$\sigma_a=\inf\{r\in\mathbb{R}\mid f \text{ converges absolutely for } \operatorname{Re}s>r\}.$$

As depicted in the diagram below, $-\infty\le\sigma_c\le\sigma_u\le\sigma_a\le\infty$. Moreover in 1913, Bohr [3] proved that $\sigma_u=\sigma_b$.

Regions of Convergence



Motivation

In 1945, Arne Beurling proposed the following problem:

Characterize all $\varphi\in L^2(0,1)$ such that $(\varphi(nx))_{n=1}^{\infty}$ is a Riesz basis in $L^2(0,1)$.

Hedenmalm, Lindqvist and Seip [1] solved the problem in 1997 using Dirichlet series: The sequence $(\varphi(nx))_n$ is a Riesz basis in $L^2(0,1)$ if and only if a certain Dirichlet series associated to φ is both bounded away from zero and infinity in the half-plane $\mathbb{C}_+=\{s\in\mathbb{C}:\operatorname{Re}s>0\}$.

The main difficulty in their approach: classify the space of multipliers of the Hilbert space $\mathcal{H}^2$; as it turns out, these are precisely the set $\mathcal{H}^\infty$ of bounded analytic functions on $\mathbb{C}_+$ which admit a Dirichlet series representation.

The Hilbert Space $\mathcal{H}^2$

We are interested in the set $\mathcal{H}^2$ of Dirichlet series $f(s)=\sum_n a_n n^{-s}$ such that $\sum_n |a_n|^2<\infty$. The set $\mathcal{H}^2$ becomes a complex Hilbert space when endowed with the inner product

$$\langle f,g\rangle=\sum_{n=1}^{\infty}a_n\bar{b}_n,$$

where $f(s)=\sum_n a_n n^{-s}$ and $g(s)=\sum_n b_n n^{-s}$. By the Cauchy-Schwarz inequality, members in $\mathcal{H}^2$ are in fact analytic functions on the half-plane $\mathbb{C}_{\frac12}=\{s\in\mathbb{C}:\operatorname{Re}s>\frac12\}$. Moreover, as proved by Hedenmalm and Saksman [2], if $\sum_n |a_n|^2<\infty$, the series

$$\sum_{n=1}^{\infty}a_n n^{-\frac12+it} \text{ converges for a.e. } t\in\mathbb{R}.$$

Finally, $\mathcal{H}^2$ is also a *reproducing kernel Hilbert space* with kernel function

$$K_{\mathcal{H}^2}(s,w)=\zeta(s+\overline{w}).$$

Hence, $\mathcal{H}^2$ is expected to play an important role in Analytic Number Theory.

The Multipliers $\mathcal{H}^\infty$

A multiplier m on $\mathcal{H}^2$ is an analytic function on the half-plane $\operatorname{Re}s>\frac12$ that has the property that $mf\in\mathcal{H}^2$ whenever $f\in\mathcal{H}^2$. The main result of [1] states that the set of all multipliers on $\mathcal{H}^2$ is $\mathcal{H}^\infty$. In view of the a.e. convergence theorem of Hedenmalm and Saksman [2], we considered the following problem:

Suppose the function $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ belongs to $\mathcal{H}^\infty$, so that the series converges on $\operatorname{Re}s>0$. Does the series then also converge a.e. on the imaginary axis?

As of now, we are unable to answer this question. However, we now describe some of the observations we have made concerning $\mathcal{H}^\infty$.

Functions Over the Polydisk

As usual, we equip $\mathcal{H}^\infty$ with the sup-norm $\|f\|_\infty=\sup_{\operatorname{Re}s>0}|f(s)|$. Consider the Dirichlet polynomial

$$f(s)=\sum_{n=1}^{N}a_n n^{-s}.$$

Let p_j represent the jth prime number, i.e., $p_1=2, p_2=3, p_3=5,\dots$ and note that there exists some $r\in\mathbb{N}$ such that $p_r\le N<p_{r+1}$. By the Fundamental Theorem of Arithmetic, for each $1\le n\le N$, there exists a unique $v=(v_1,v_2,\dots,v_r)$ such that $n=p_1^{v_1}p_2^{v_2}\cdots p_r^{v_r}$. Set $c_v=a_n$ and $z_j=p_j^{-s}$ for each $1\le j\le r$. Note that $n^{-s}=p_1^{-sv_1}p_2^{-sv_2}\cdots p_r^{-sv_r}=z_1^{v_1}z_2^{v_2}\cdots z_r^{v_r}$. Thus if we put $z=(z_1,z_2,\dots,z_r)$, $f(s)$ formally takes the form

$$F(z)=\sum_v c_v z^v.$$

As $|z_j|\le 1$, F is considered as defined over the whole closed polydisk $\overline{\mathbb{D}}^r$. Moreover, by an application of Kronecker's Approximation Theorem and the Maximum Modulus Principle, $\|f\|_\infty=\|F\|_\infty$.

A Theorem on Dirichlet Polynomials

If $f(s)=\sum_{n=1}^{N}a_n n^{it}$, then $\sum_{p\le N}|a_p|\le\|f\|_\infty$, where the sum is taken over prime numbers p.

Proof. Consider $F(z)=\sum_v c_v z^v$ and let $F_k(z)$ be the sum of the terms $c_v z^v$ such that $|v|=v_1+v_2+\cdots+v_r=k$. Note that $F_k(z)$ is homogeneous of degree k, i.e., $F_k(\lambda z)=\lambda^k F_k(z)$ for all $\lambda\in\mathbb{C}$. Since $F(z)=\sum_{k=0}^{\infty}F_k(z)$,

$$F(ze^{i\theta})=\sum_{k=0}^{\infty}F_k(ze^{i\theta})=\sum_{k=0}^{\infty}F_k(z)e^{ik\theta}.$$

Therefore for $m\in\mathbb{N}_0$,

$$\frac{1}{2\pi}\int_0^{2\pi}F(ze^{i\theta})e^{-im\theta}d\theta=\frac{1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi}F_k(z)e^{i(k-m)\theta}d\theta=F_m(z)$$

and hence

$$|F_m(z)|\le\frac{1}{2\pi}\int_0^{2\pi}|F(ze^{i\theta})|d\theta=\|F\|_\infty.$$

In particular, $\left|\sum_{p\le N}a_p z_p\right|=|F_1(z)|\le\|F\|_\infty$. When $a_p\ne0$, we may choose $z_p=\frac{\bar{a}_p}{|a_p|}$ and obtain

$$\sum_{p\le N}|a_p|=|\sum_{p\le N}a_p z_p|\le\|f\|_\infty. \qquad \square$$

Corollary. $\sum_{p\text{ prime}}|a_p|\le\|f\|_\infty$ whenever $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ belongs to $\mathcal{H}^\infty$.

An Interesting Question

For $f(s)=\sum_n a_n n^{-s}$ in $\mathcal{H}^\infty$, we see that $\sum_{p\text{ prime}}|a_p|\le\|f\|_\infty\le\sum_{n=1}^{\infty}|a_n|$.

Thus, we ask:

If $f\in\mathcal{H}^\infty$, must $\sum_{n=1}^{\infty}|a_n|<\infty$? If so, is there an estimate of the form $\sum_{n=1}^{\infty}|a_n|\le C\|f\|_\infty$ for some constant C independent of f?

References

[1] Hedenmalm, H., Lindqvist, P., Seip, K.: A Hilbert space of Dirichlet series and a system of dilated functions in $L^2(0,1)$. Duke Math. J. **86**, 1-36 (1997)

[2] H. Hedenmalm, E. Saksman, Carleson's convergence theorem for Dirichlet series, Pacific J. Math. **208** (2003), 85-109.

[3] Bohr, H.: Über die gleichmässige Konvergenz Dirichletscher Reihen. J. Reine Angew. Math. **143**, 203-211 (1913)