First-kind Volterra Equations

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Abstract

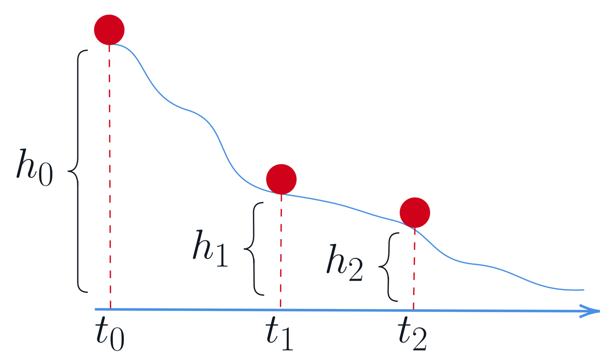
First-kind Volterra equations arise in various applications, e.g., in models of population dynamics and in the inverse heat conduction problem. Roughly, these equations have the form

$$\int_0^t k(t-s)u(s)ds = f(t) \quad \text{for} \quad t \in [0,1]$$

where k and f are given and u is an unknown function.

Problem

When an object is pushed down a slide, the object follows a continuous path. As the object descends, the object's location (i.e. height) can be modeled as a function of time.



We are interested in the *inverse problem*, that is, given a function f that specifies the total time of descent (for a given starting height), find an equation of the curve u that yields this result. This inverse problem is known as *Abel's mechanical problem*; the relationship between f and u is (up to multiplicative constants) given by the integral equation

$$\int_0^t \frac{1}{\sqrt{t-s}} u(s)\, ds = f(t) \qquad (1)$$

More generally, many other problems can be expressed in the form

$$\int_0^t k(t-s)u(s)ds = f(t) \quad for \quad t \in [0,1]$$

for some function k called the Kernel. For simplicity Equation 1 can be written as

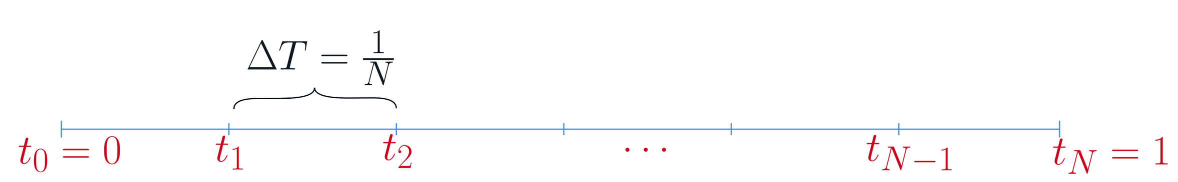
$$\mathcal{A}u = f$$

In reality, only a discrete set of data is available. (e.g. desired time for an object to reach a concrete height). However, one seeks to reconstruct a continuous function. This has led to the development of various methods such as *Collocation* and *Projection*. Our problem consists to find a function u that solves

$$Au = f$$

if only a finite set of values of f are known, say on $[0,1]$. For simplicity, we assume $f(t_1), f(t_2), \ldots, f(t_N)$ are known

Collocation



For $i \geq 2$, we define the characteristic functions $\chi_i(t) = 1$ if $t \in (t_{i-1}, t_i]$, and $\chi_i(t) = 0$ otherwise. Also we define $\chi_1(t) = 1$ if $t \in [0, t_1]$ and $\chi_1(t) = 0$ otherwise. In the method of **Collocation**, we seek a piecewise constant function $u = \sum_{k=1}^{N} c_k \chi_k$ so that

$$(\mathcal{A}u)(t_i) = f(t_i) \quad \text{for } i = 1, \ldots, N \qquad (2)$$

Matrix Representation

After making change of variables it can be shown that

$$Au(t_j) = \sum_{i=1}^{j} c_i \int_0^{\Delta T} k((j-i+1)\Delta T - s)ds$$

Therefore, the collocation equations $(Au)(t_i) = f(t_i)$ can be rewritten, in matrix form, as:

$$\begin{bmatrix} \int_0^{\Delta T} k(1\Delta T - s)ds & 0 & \cdots & 0 \\ \int_0^{\Delta T} k(2\Delta T - s)ds & \int_0^{\Delta T} k(1\Delta T - s)ds & \cdots & 0 \\ & & \cdots & \vdots \\ \vdots & & \cdots & 0 \\ & & \ddots & 0 \\ \int_0^{\Delta T} k(N\Delta T - s)ds & \int_0^{\Delta T} k((N-1)\Delta T - s)ds & & \int_0^{\Delta T} k(1\Delta T - s)ds \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ \vdots \\ \vdots \\ f(t_N) \end{bmatrix}$$

It can be seen that this equation has a unique solution.

Approximations

A mathlab code was created to show approximations for the function $u(t) = \cos(4\pi t)$ with kernel $k(t-s) = t^{\nu}$ and for different values of N and ν. We also introduce noise to show how small perturbations in our data can affect the approximation to the function u.

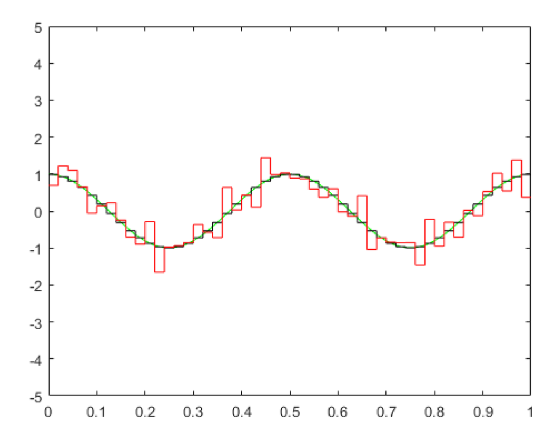


Figure 1: $N = 50,\ \nu = 1,\ err = 0.01$

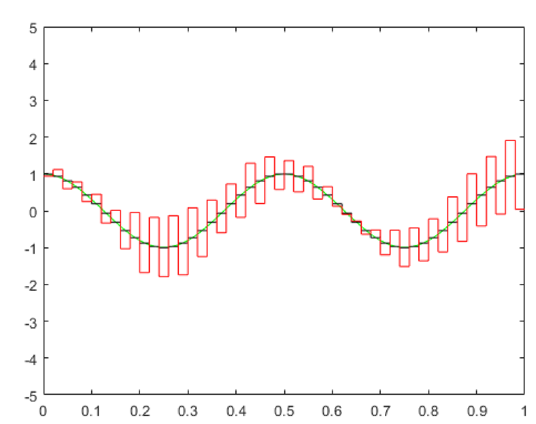


Figure 2: $N = 50,\ \nu = 2,\ err = 0.001$

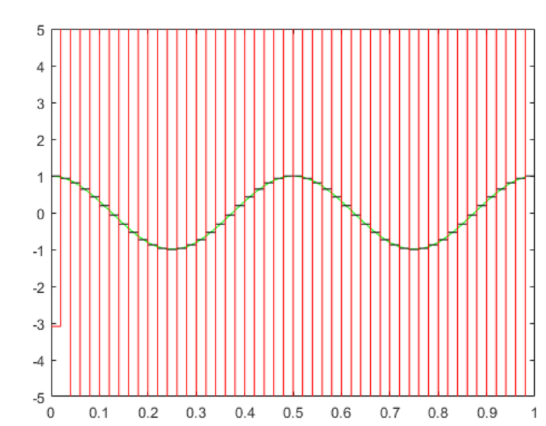


Figure 3: $N = 50,\ \nu = 2,\ err = 0.01$

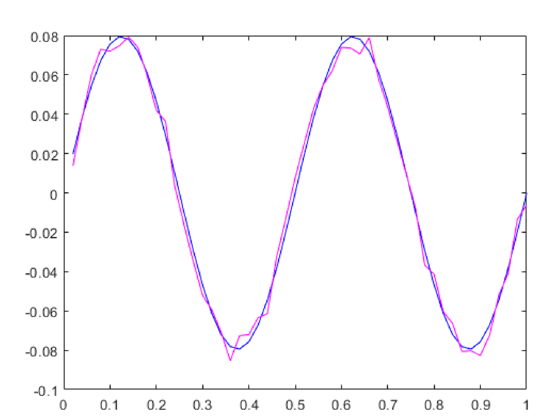


Figure 4: Data with error 1%

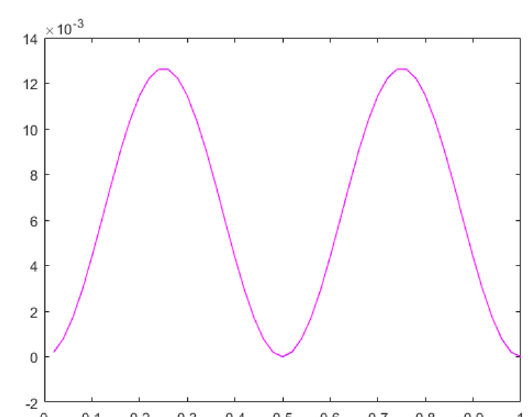


Figure 5: Data with error 0.1%

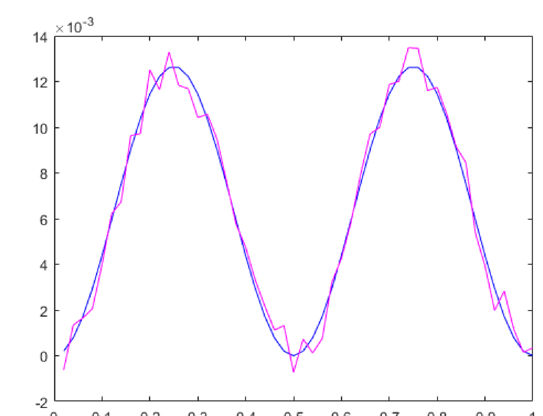


Figure 6: Data with error 1%

Projection

The method of **Projection** for A seeks to approximate a solution to the equation $Ax = y$ by a sequence $(x_n)_{n=1}^{\infty}$ of solutions to the equation

$$P_N A P_N x = P_N y, \quad N = 1, 2, \ldots \qquad (3)$$

Since our goal is to solve $Au = f$, then Equation 3 in our context can be written as

$$P_N A P_N u = P_N f \text{ for } \alpha \geq 1 \text{ and } N = 1, 2, \ldots \qquad (4)$$

where $u, f \in L^2[0,1]$ and A is the integral Volterra operator.

Matrix Representation

By cases and after a change of variables, it can be shown that

$$P_N(A\chi_i) = \frac{1}{N}\Big(\sum_{j=i}^{N} a_{i,j}\chi_j - \sum_{j=i+1}^{N} b_{i,j}\chi_j\Big)$$

where

$$a_{i,j} = \int_{t_{j-1}}^{t_j} \Big(\int_0^{t-t_{i-1}} k(\xi)d\xi\Big)dt \text{ and } b_{i,j} = \int_{t_{j-1}}^{t_j} \Big(\int_0^{t-t_i} k(\xi)d\xi\Big)dt$$

Collocation vs Projection

The following graphs show the roots of the calculus polynomial resultant from the method of *Collocation* and *Projection* indicating the instability of both methods as ν become larger.

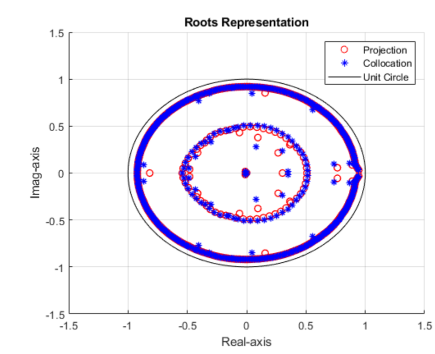


Figure 7: $N = 500,\ \nu = 20$

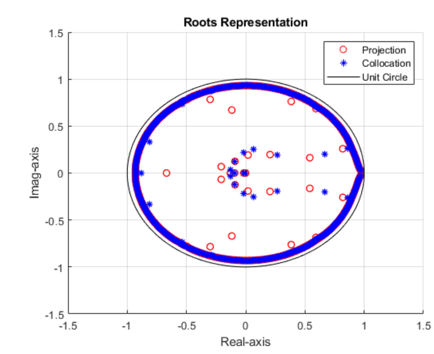


Figure 8: $N = 500,\ \nu = 10$

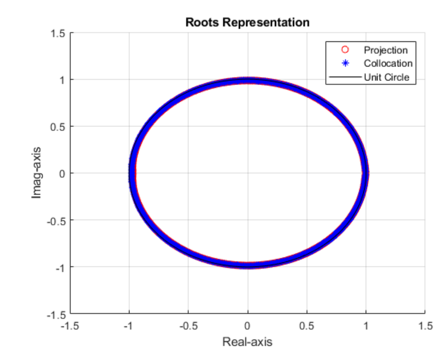


Figure 9: $N = 500,\ \nu = 1$

Conclusion

As we can see from the graphs, given the kernel $k(t-s) = t^{\nu}$ and values of $\nu \geq 2$, the method of *Collocation* by piece-wise constant functions and *Projection* are unstable under small perturbation. We tested the methods for various kernels as well, and the results were similar. The results make one think in a different approach to solving the problem, that is, to represent it as a second kind of Volterra equation. The first kind of Volterra equation is known to lead to difficulties, as shown here. Whereas in the second kind of Volterra equation, these difficulties are nonexistent.

Acknowledgements

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