Singular Values of the Volterra Operator

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Abstract

This research focuses on singular values of the Volterra operator, V, a problem that has been studied extensively. It requires finding the eigenvalues of the self-adjoint, positive, compact operator of the form

$$(V^*Vf)(x) = \int_x^1 \int_0^t f(s)dsdt,$$

which results in solving a second order differential equation. We then explore more complicated versions of this problem by taking powers of the Volterra operator. This leads to more complex, higher-order differential equations to establish a generalized formula for the system of equations given by boundary conditions. Additionally, non-interger powers of the operator, V^α, are also examined, specifically $\alpha \in (0,1)$.

Introduction

The Volterra operator, V, maps $L^2[0,1]$ to $L^2[0,1]$ by

$$(Vf)(x) = \int_0^x f(t)dt.$$

In order to understand singular values we must first define a few key concepts for our operator.

Definition (Adjoint)

Consider the bounded operator A on a Hilbert space H. The adjoint operator $A^* : H \to H$ is defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \forall x, y \in H.$$

The Volterra operator has an inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

This leads to the adjoint of the Volterra operator to be

$$(V^*f)(x) = \int_x^1 f(t)dt.$$

Definition (Singular Values)

Given an operator A, s_j is a singular value of A if it belongs to the set

$$s_j(A) = \{\sqrt{\lambda} | \ \lambda \text{ is an eigenvalue for } A^*A\}.$$

Solving the Initial Case

To find the singular values, s, for the Volterra operator in its simplest form requires solving

$$(V^*Vf)(x) = s^2 f(x) \tag{1}$$

$$\int_x^1 \int_0^t f(s)dsdt = s^2 f(x). \tag{2}$$

Differentiating both sides twice results in the differential equation

$$f''(x) + \frac{1}{s^2} f(x) = 0. \tag{3}$$

Letting $\omega = 1/s$ and solving this simple differential equation leads to

$$f(x) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}, \quad \text{and}$$
$$f'(x) = i\omega c_1 e^{i\omega x} - i\omega c_2 e^{-i\omega x}.$$

To solve for our unknown constants we refer back to the original equation 2. Our first integral is evaluated from x to 1, and hence if $x = 1$, the integral is zero. This implies that either s^2 is zero or $f(1)$ is zero. However, in equation 3 we divided by s^2, implying it cannot equal zero. Thus, $f(1) = 0$. By a similar argument we can see that $f'(0) = 0$. These boundary conditions show that $c_1 = c_2$. Additionally, we find that

$$f(x) = 2c_1 \cos(\omega x).$$

More importantly, as $f(1) = 0$, we find that ω must be of the form $((2k+1)\pi)/2$. Therefore,

$$s = \frac{2}{(2k+1)\pi}, \ k \in \mathbb{Z}.$$

Integer Powers of the Volterra Operator

A more complicated problem arises for finding singular values of the powers of the Volterra operator. We define $V^n f$ by

$$V^{n+1}f = V(V^n f), \quad n \in \mathbb{N}.$$

Using induction, it can be shown that

$$(V^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt, \quad \text{and}$$
$$((V^*)^n f)(x) = \frac{1}{(n-1)!}\int_x^1 (t-x)^{n-1} f(t)dt.$$

In order to find the singular values for the power of the operator, we now need to solve

$$((V^*)^n V^n f)(x) = s^2 f(x), \text{ or equivalently}$$

$$\int_x^1 \frac{(t-x)^{n-1}}{(n-1)!} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)dsdt = s^2 f(x).$$

However, this also can be evaluated as a differential equation, this time by taking the derivative $2n$ times, resulting in

$$(-1)^n f = s^2 f^{(2n)}$$

Additionally, we find that the boundary conditions are

$$f(1) = f'(1) = ...f^{(n-1)}(1) = f^{(n)}(0) = ... = f^{(2n-1)}(0) = 0.$$

If we let $\omega = 1/s^{1/n}$, then we need to solve the differential equation

$$f^{(2n)} - (i\omega)^{2n} f = 0.$$

Consider the characteristic equation for the previous differential equation.

$$z^{(2n)} - (i\omega)^{(2n)} = 0$$
$$z^{(2n)} = (i\omega)^{(2n)}.$$

We can rewrite this as

$$r^{(2n)} e^{2i\theta n} = z^{(2n)} = (i\omega)^{(2n)}.$$

Then $r^{(2n)} = \omega^{(2n)}$ and $e^{2i\theta n} = i^{(2n)}$. This implies that $\theta = \frac{\pi}{2} + \frac{\pi k}{2}$ for $k = 1, 2, ..., 2n$. Thus,

$$z = \omega e^{i(\pi/2 + (\pi k)/n)} = \omega e^{i\pi/2} e^{(\pi/n)^k} = i\omega e^{(i\pi/n)^k}.$$

To find our eigenfunction, we first let $\xi_0 = e^{i\pi/n}$. Then,

$$f(x) = c_1 e^{i\omega\xi_0} + c_2 e^{i\omega\xi_0^2} + ... + c_{2n} e^{i\omega\xi_0^{2n}}$$

Coefficient Matrix

Using the boundary conditions and eigenfunction found previously, we can create a coefficient matrix.

$$\begin{bmatrix} e^{\omega i\xi_0} & e^{\omega i\xi_0^2} & \dots & e^{\omega i\xi_0^{2n-1}} & e^{\omega i\xi_0^{2n}} \\ e^{\omega i\xi_0} & \xi_0 e^{\omega i\xi_0^2} & \dots & \xi_0^{2n-2} e^{\omega i\xi_0^{2n-1}} & \xi_0^{2n-1} e^{\omega i\xi_0^{2n}} \\ e^{\omega i\xi_0} & \xi_0^2 e^{\omega i\xi_0^2} & \dots & \left(\xi_0^{2n-2}\right)^2 e^{\omega i\xi_0^{2n-1}} & \left(\xi_0^{2n-1}\right)^2 e^{\omega i\xi_0^{2n}} \\ \vdots & & \ddots & & \\ e^{\omega i\xi_0} & \xi_0^{n-1} e^{\omega i\xi_0^2} & \dots & \left(\xi_0^{2n-2}\right)^{n-1} e^{\omega i(\xi_0)^{2n-1}} & \left(\xi_0^{2n-1}\right)^{n-1} e^{\omega i(\xi_0)^{2n}} \\ 1 & \xi_0^n & \dots & \left(\xi_0^{2n-2}\right)^n & \left(\xi_0^{2n-1}\right)^n \\ \vdots & & \ddots & & \\ 1 & \xi_0^{2n-1} & \dots & \left(\xi_0^{2n-2}\right)^{2n-1} & \left(\xi_0^{2n-1}\right)^{2n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ c_{n+1} \\ \vdots \\ c_{2n} \end{bmatrix}$$

Case when $n = 2$

For the specific case when $n = 2$, the boundary conditions lead to the following 4×4 matrix for the coefficients of the terms.

$$\begin{bmatrix} e^{\omega} & e^{-\omega} & e^{i\omega} & e^{-i\omega} \\ e^{\omega} & -e^{-\omega} & ie^{i\omega} & -ie^{-i\omega} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} f(1) \\ f'(1) \\ f''(0) \\ f'''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we find the determinant of this matrix and set it equal to zero, we can identify our singular values. This results in

$$\cos(t)\cosh(t) + 1 = 0,$$

where the singular values are equivalent to t^{-2}.

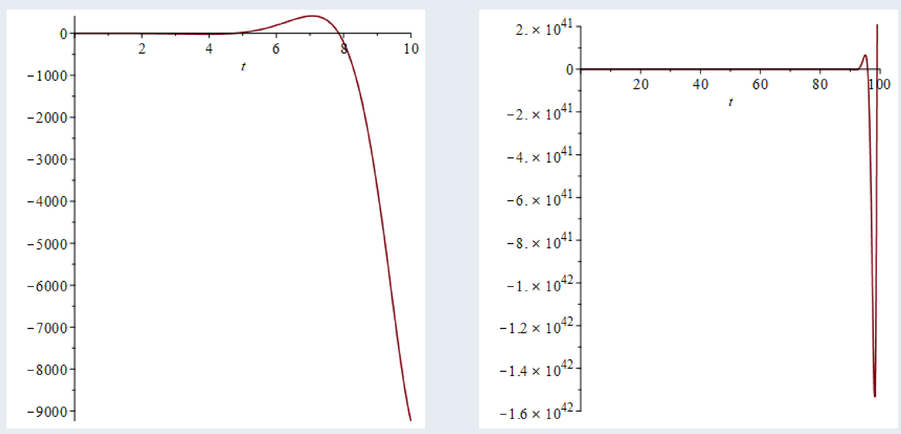

Figure 1: $f(t) = \cos(t)\cosh(t) + 1$

Non-Integer Powers of the Volterra Operator

We now consider powers of the operator that are not integers, namely the power $\alpha \in (0,1)$. Here, we define the operator to the power of α by

$$(V^\alpha f)(x) = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt.$$

This leads to finding the eigenvalues of

$$((V^*)^\alpha V^\alpha f)(x) = \int_x^1 \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dsdt.$$

We can immediately observe that taking derivatives does not apply in this case. Instead, we focused on the range of V^α, $((V^*)^\alpha$, and finally $((V^*)^\alpha V^\alpha$. To find these, we take advantage of the semi-group property

$$V^\alpha V^\beta = V^{\alpha+\beta}.$$

To start, we consider any g such that $V^\alpha f = g$. Then,

$$g = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt.$$

Thus, if we are evaluating at $(V^\alpha f)(0)$, the limits of integration go from 0 to 0, and therefore $g(0) = 0$. Additionally, given D as the differential operator,

$$f = D(V^{1-\alpha}g).$$

As $f \in L^2[0,1]$, then $D(V^{1-\alpha}g) \in L^2[0,1]$. Furthermore, for us to be able to differentiate, we need $V^{1-\alpha}g \in AC[0,1]$. Putting this altogether, we have

$$Ran(V^\alpha) = \{g : g(0) = 0, (V^{1-\alpha}g)' \in L^2[0,1], V^{1-\alpha}g \in AC[0,1]\}.$$

By a completely analogous approach, we find that

$$RanV^{\alpha*} = \{h : h(1) = 0, (V^{(1-\alpha)*}h)' \in L^2[0,1], V^{(1-\alpha)*}h \in AC[0,1]\}.$$

For $(V^\alpha)^* V^\alpha$, we note that this is a map that takes $Ran(V^\alpha) \to Ran(V^{\alpha*})$. Here, we are applying $V^{\alpha*}$ to $Ran(V^\alpha)$. If we let $V^{\alpha*}V^\alpha f = k$, then

$$Ran(V^{\alpha*}V^\alpha) = \{k : k(1) = 0, (V^{(1-\alpha)*}k)' \in L^2[0,1],$$
$$V^{(1-\alpha)*}k \in AC[0,1], -D(V^{(1-\alpha)*}k) = g\}.$$

Conclusion

While much progress has been made towards finding the singular values of the Volterra operator, the complexity of the problem ensures future work will be needed.

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