Saddle Points of Analytic Matrix-Valued Functions

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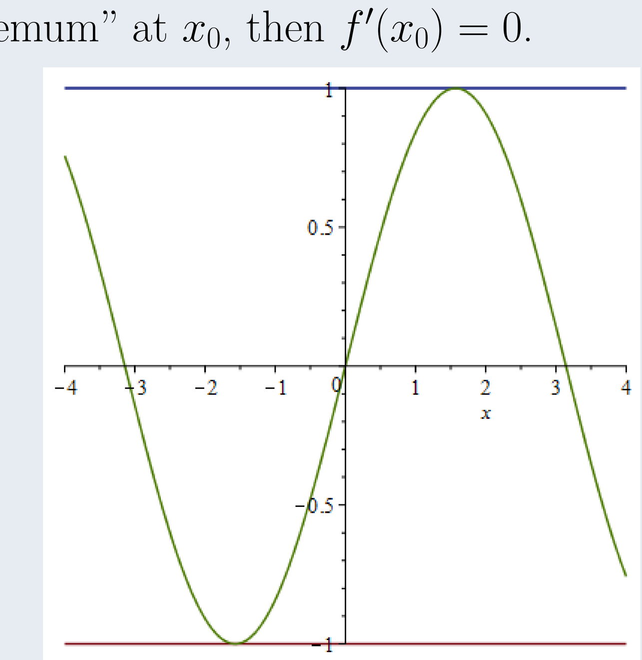
Abstract

- Optimization is one of the things that calculus gets applied to the most. We want to know where a function is at its maximum or minimum value. When at a maximum or minimum of a function, the derivative at that point is zero. Sometimes, when the derivative is zero, it is neither a max nor a min. When the derivative is zero and the point is neither a max nor a min, it is referred to as a saddle point. Potential saddle points may be narrowed down in real-valued functions by use of the Hessian matrix. Less is known about $n \times n$ complex-analytic matrix-valued functions. Except for the 1×1 case, where a point is a saddle if it is also a saddle of the modulus of the function, and where being a critical point guarantees being a saddle point when not on the boundary. In order to determine the conditions for a saddle point existing with matrix-valued functions we utilize Operator and Frobenius norms. [4]

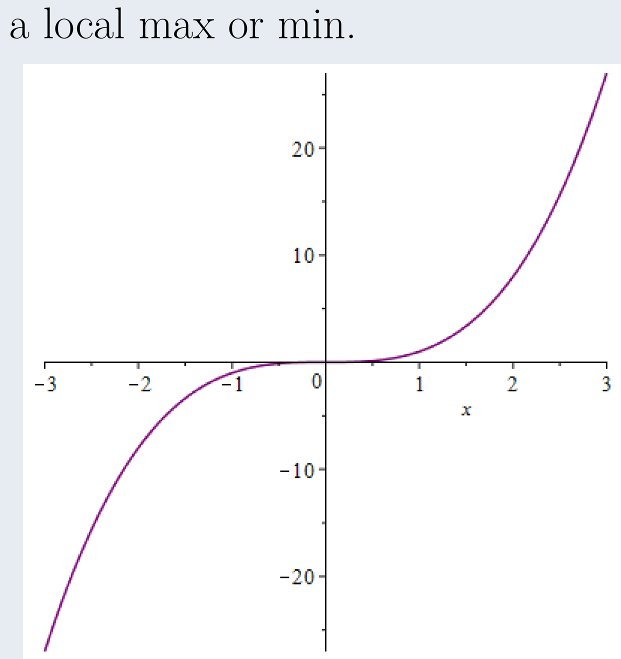
Fermat's Interior Extremum Theorem

In order to understand critical points at a matrix-valued level let us first understand them at the real-valued level. Pierre De Fermat granted us this theorem back in the 1600s!

Fermat's Theorem of Stationary Points: Suppose x_0 is an interior point of an open interval I and f is a real-valued function defined on I which is differentiable at x_0. If f has a "local extremum" at x_0, then $f'(x_0) = 0$.



In contrast, a **saddle** is a point, x_0, where $f'(x_0) = 0$, but $f(x_0)$ is not a local max or min.



[3]

Complex Functions Have No Order

We wish to analyze complex-valued functions because they are essentially the 1×1 case of matrix-valued functions. In the beginning of the proof to Fermat's Interior Extremum Theorem, you first assume that a point is either a maxima or a minima. Call that point a and you get

$$f(a) \geq f(a+k)$$

if a is a maxima and if k is some real number. Such inequalities cannot hold for complex variables. The reason is due to axioms related to natural ordering: $\forall\ a, b, c$

1. $a \leq b$ implies $a + c \leq b + c$
2. $0 \leq a$ and $0 \leq b$ implies $0 \leq ab$

Assume that complex numbers do obey these rules, then either $i \geq 0$ or $i \leq 0$.
If $i \geq 0$, then by the second rule we can say that

$$i * i \geq 0$$

but we know that $i * i = -1$, and to say $-1 \geq 0$ is a known falsehood. We could also use rule one, and since we know $1 \geq 0$ is true, we can say

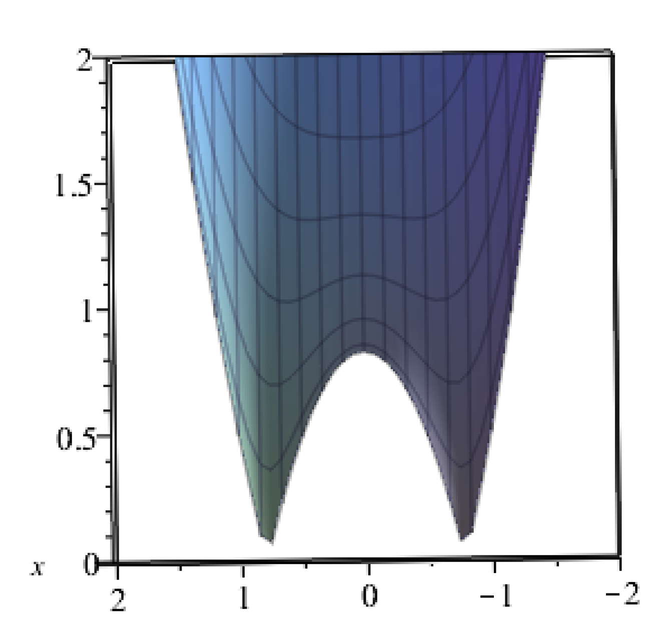
$$i * i + 1 \geq 0 + 1$$

therefore we arrive at $0 \geq 1$ which we know is false. If $i \leq 0$, we can say $-i \geq 0$ so

$$(-i)(-i) \geq 0$$

We get $-1 \geq 0$ again which we know isn't true. So, complex numbers can not be ordered! We circumvent this problem utilizing the modulus of the complex function.

Saddle Points with respect to the Modulus



This is a plot of $|z^2 + e^z|$, the modulus of $z^2 + e^z$, which has a saddle point at $z = 0$. For a complex number $a + bi$, the modulus is $\sqrt{a^2 + b^2}$. Since a and b are both real, the modulus is also real, and non-negative. We say $f(x)$ has a saddle if $|f(x)|$ has a saddle.

The Bak-Newman-Ding Theorem

Since the problem was lack of ordering for complex functions, and the modulus fixes that problem, the natural question to ask is "do complex functions have similar conditions for saddle points as their real counterparts?". The answer to that question is a resounding NO. This is well known and flows from the maximum and minimum "modulus theorems".

- By the maximum modulus theorem, $f(z)$ has no local extrema except for zeros, if it has any (the zeros are where $|f(z)|$ attains its absolute minimum)
- From the max & min modulus theorems, if $f(z)$ is analytic in an open set containing a compact set K, and is not constant, then the max & min modulus $|f(z)|$ over K is found on the boundary of K.
- **Bak-Newman-Ding Theorem:** Let f be a nonconstant complex analytic function on an open set Ω, $z_0 \in \Omega$ is a saddle point of an analytic function f if and only if $f'(z_0) = 0$ and $f(z_0) \neq 0$.

[1]

Square Matrix-Valued Functions

In order to try to generalize the Bak-Newman-Ding theorem to Square MVF's (Matrix-Valued Functions), we must first define the MVF as simply a matrix with complex-valued functions as the entries.

Example: If

$\mathbb{F} = \begin{bmatrix} z & 0 \\ 0 & z^2 \end{bmatrix}$, then F(-1)=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, F(1+i)=$\begin{bmatrix} 1+i & 0 \\ 0 & 2i \end{bmatrix}$

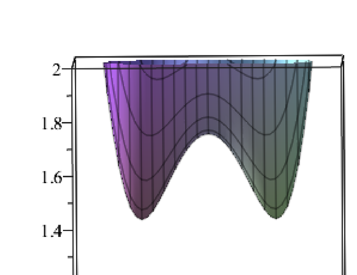
The 2×2 Case

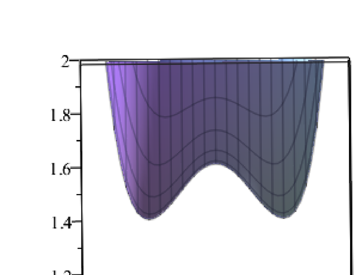
The simplest case after understanding the 1×1 case is the 2×2 case.

- Again, we have to deal with a lack of natural ordering, and with a 2×2 matrix we are mapping two variables ($\Re(z)$ and $\Im(z)$) to eight other variables ($4 \in \Re(z)$, and $4 \in \Im(z)$).
- We use "norms" to tackle this problem as we have used the modulus before.
- There are multiple kinds of norms, for this project only the Frobenius/Euclidean norm and Operator norm are used.
- Frobenius/Euclidean norm:
  $||A||_F = \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2}$
- Operator norm:
  $$||A|| = \max\left\{\sqrt{|a_{11}x_1 + a_{12}x_2|^2 + |a_{21}x_1 + a_{22}x_2|^2} : x_1^2 + x_2^2 = 1\right\}$$

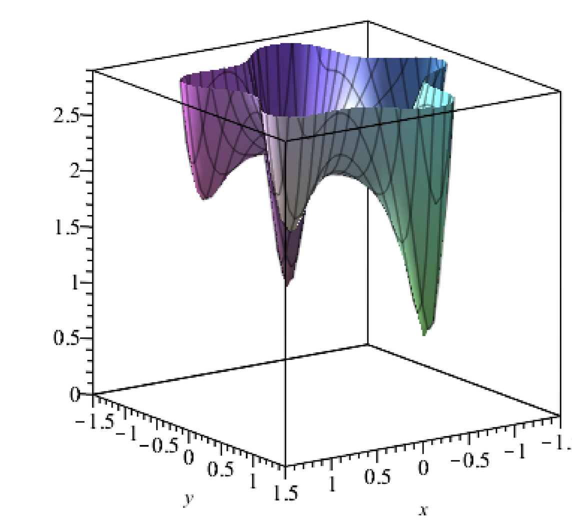
In the following example we see that whether the operator (right) or frobenius (left) norm is used, the property of having a saddle point is preserved. Hence, we may define the saddle of an MVF as the saddle of either its frobenius or operator norm. [2]

$$\mathbb{F}(z) = \begin{bmatrix} z^2+1 & 1 \\ 0 & 1 \end{bmatrix}$$





Current Progress



The norm of $\mathbb{F}(z) = \begin{bmatrix} z^2+1 & 0 \\ 0 & z^5+2 \end{bmatrix}$, generated in Maple above, appears to have 4 saddles, if you could see it from all sides. This is one less than the degree of the highest degree polynomial out of all of the elements in the matrix, so I would like to generalize and say that the number of saddles n is

$$n = (\max \deg a_{ij}) - 1$$

. The problem with that is a casual observer may not simply "eyeball" saddle points. Analytical rigor requires more stringent testing techniques to determine if a point is a saddle or not. Modern techniques include the use of the Hessian matrix. The Hessian is a square matrix of second-order partial derivatives. Applying this technique may yield a greater number of potential saddle points than my formula suggests.

Open Questions

- Is there an analog to the Bak-Newman-Ding theorem for MVF's?
- If a non-constant function has 2 local min's and no local max, then must it have a saddle in between the min's?
- How does one analytically prove that a point is a saddle for an MVF?

References

[1] Joseph Bak, Pisheng Ding, and Donald Newman. Extremal points, critical points, and saddle points of analytic functions. *The American Mathematical Monthly*, 114(6):540–546, 2007.

[2] Alberto A. Condori. Maximum principles for matrix-valued analytic functions, 2019.

[3] M. Sullivan and K. Miranda. *Calculus: Early Transcendentals*. W. H. Freeman, 2018.

[4] W.R. Wade. *Introduction to Analysis, An,*. Pearson Education, 2014.

Acknowledgements

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